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STRUCTURE OF SHOCK WAVES IN ELASTOPLASTIC RELAXING MEDIA

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At the present time shock and explosive loads are being more and more widely used in various technical processes. In this case, an adequate description both of the process of the propagation of a shock wave and of the change in the medium as a result of the shock action is of great importance.

Elastoplastic waves have been discussed earlier in a number of pieces of work [1-3], taking account of the behavior of dislocations at the front of the shock wave. In [1] a model was developed for the description of the inelastic behavior of iron and low-carbon steel in a wide range of change in the deformation rates. A solution is given to the problem of the plane collision of plates. In [2], along with a numerical solution of the problem of the porpagation of an elastoplastic wave, a stationary wave is discussed. It is shown that the front of a shock wave has a multiwave structure. However, as an expression for the velocity of the dislocations the authors of [2] used only an exponential dependence on the intensity of the tangential stresses and did not consider the important case of a power dependence. In [3], on the basis of the dynamics of dislocations, the theory of a fully established wave profile is discussed; numerically calculated profiles are compared with experimental profiles, obtained by the methods of laser interferometry. It is shown that the velocity of the dislocations with the shock-wave compression of aluminum is well described by a power dependence. In addition, it is shown that for aluminum the density of mobile dislocations increases linearly with a rise in the value of the plastic shear γ_p.

In the present article the question of the structure of the waves of the load in elastoplastic media is discussed; a dislocation model of the dynamic plasticity is used [4-6]. Within the framework of this model, it is possible to describe not only the dynamics of the plastic deformation, but also to consider the structural changes which take place in a material under the action of dynamic loads.

The analogous problem of the structure of a shock wave, using a phenomenological approach to a description of the relaxation of the tangential stresses, was discussed in [7]; however, in this article there was no detailed discussion of the role of the nonlinearity of the process of the relaxation of the stresses, and effects connected with the change in the density of the dislocations were not taken into consideration.

Let us consider a shock wave, whose width Δ is small in comparison with the curvature of the front and the distance at which appreciable damping of the shock wave takes place. In this case, the structure of the wave will be determined by the solution of the steady-state plane problem [8].

Going over to a moving system of coordinates in which the front is motionless, the equations of motion can be written in the form [7]

$$\rho u = \rho_0 u_0,\ \sigma_1-\sigma_{10} = (\rho_0 u_0)^2(1/\rho-1/\rho_0), \tag{1}$$

where ρ is the density; u is the velocity; σ_1 is the stress along the axis of propagation; ρ_0, u_0, σ_{10} are the corresponding values ahead of the front.

We shall consider not-too-strong shock waves, so that the temperature behind the front of the wave does not exceed the melting temperature. In this case, the thermal components of the pressure can be neglected [8] and the equation of state can be written in the form

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$$p = p(\rho),\ p = -(\sigma_1 + 2\sigma_2)/3, \tag{2}$$

where p is the pressure; σ_2 is the stress in a direction perpendicular to the direction of propagation of the wave.

In addition to the equation of state, in a solid body the connection between the tangential stresses $\tau = \sigma_1 - \sigma_2$ and the deformations must also be given. As relationships determining the tangential stresses we take the equations of the dislocation model of the dynamic plasticity [4-6] which, in our case, assume the form

$$-\frac{\rho_0 u_0}{\rho}\frac{d\rho}{dx} = \frac{\rho_0 u_0}{2G\rho}\frac{d\tau}{dx} + \mu b N_m v(\tau), \tag{3}$$

where G is the elastic modulus; μ is the modulus of the orientation tensor; b is the Burgers vector; N_m is the number of mobile dislocations per unit of surface; $v(\tau)$ is the mean velocity of the dislocations under the action of the tangential stress. The velocity of the dislocations $v(\tau)$ depends on the temperature. However, for large velocities, this dependence has been insufficiently well investigated [9] and will not be taken into consideration in what follows.

The number of mobile dislocations N_m varies during the process of plastic deformation, which is connected both with the multiplication and the fixing of the dislocations. This change can be connected with the absolute value of the plastic shear [6] γ_p:

$$\frac{\rho_0 u_0}{\rho}\frac{d\gamma_p}{dx} = \mu b N_m(\gamma_p)\,|v(\tau)|. \tag{4}$$

This relationship is important not only for determination of the profile of the wave, but also for predicting the changes in the deformation properties of the material as the result of shock action.

The system of equations (1)-(4) can be solved with respect to the density ρ

$$\frac{(\rho c_l)^2 - (\rho_0 u_0)^2}{\rho^3(c_l^2 - c^2)}\frac{d\rho}{dx} = -\mu b N_m(\gamma_p)\frac{\rho v(\tau)}{\rho_0 u_0}, \tag{5}$$

$$\tau = \frac{3}{2}\left[p - (\rho_0 u_0)^2 \frac{\rho - \rho_0}{\rho\rho_0}\right] \quad (\sigma_{10} = 0);$$

$$\frac{d\gamma_p}{dx} = \frac{(\rho c_l)^2 - (\rho_0 u_0)^2}{\rho^3(c_l^2 - c^2)}\frac{d\rho}{dx}, \tag{6}$$

where $c^2 = dp/d\rho$; $c_l^2 = c^2 + 4G/3\rho$ are the volumetric and longitudinal velocities of sound, which, generally speaking, depend on the density.

To determine the character of the solution of Eq. (5), let us examine the behavior of the coefficients of this equation in the plane $\{\rho, u_0\}$. Here we assume that the velocity of the dislocations $v(\tau)$ reverts to zero with $\tau = 0$, and $N_m > 0$. The lines determined by the equations

$$\rho = \rho_0 u_0 / c_l; \tag{7}$$

$$p(\rho) - \rho_0 u_0^2\left(1 - \frac{\rho_0}{\rho}\right) = 0, \tag{8}$$

are lines at which the following conditions are satisfied: The coefficient with $d\rho/dx$ reverts to zero and the right-hand part of Eq. (5) reverts to zero correspondingly.

These lines are shown schematically in Fig. 1, where curve 1 corresponds to the solution of Eq. (7). Equation (8) has one trivial solution $\rho = \rho_0$, and the second solution is shown in Fig. 1, curve 2. These lines

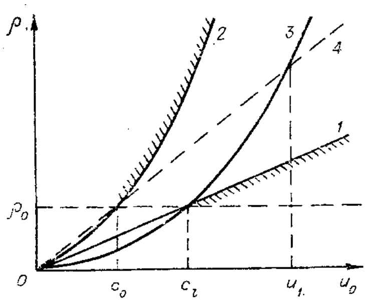

Fig. 1

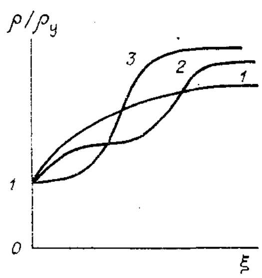

Fig. 2

divide the plane $\{\rho, u_0\}$ into parts with different signs of the derivative $d\rho/dx$. The region of positive values of $d\rho/dx$, corresponding to the waves of the load, is not hatched in Fig. 1.

As can be seen from Fig. 1, with $c < u_0 < c_l$ a continuous wave of the load can exist. With $u_0 > c_l$, the solutions must be sought in the class of discontinuous functions. It follows from Eq. (3) that the discontinuity in the solution must be elastic. The conditions at the shock wave are given by the relationships

$$\rho \frac{d\tau}{d\rho} = -2G, \quad \tau = \frac{3}{2}\left[p(\rho) - \rho_0 u_0^2\left(1 - \frac{\rho_0}{\rho}\right)\right],$$

which follow from the conditions that there is no plastic deformation at the elastic discontinuity. The line of the elastic shock wave is shown in Fig. 1 by curve 3. It lies in the region of positive values of $d\rho/dx$. Thus, with $u_0 > c_l$ the shock front consists of an elastic shock wave, behind which follows a relaxation part. With the solution of Eqs. (5) and (6), it must be taken into consideration that, at an elastic discontinuity, $\gamma_p = 0$. In the relaxation part of the profile there is a further increase in the density, until it reaches a line determined by Eq. (8), at which $\tau = 0$. As follows from the condition $\tau = 0$, this line corresponds to the hydrodynamic Hugoniot adiabatic.

With $c < u_0 < c_l$, continuous stationary waves can exist. To understand their physical meaning, let us examine (5) with $\rho \to \rho_0$:

$$u_0 \frac{c_l^2 - u_0^2}{\rho_0 (c_l^2 - c^2)} \frac{d\rho}{dx} = -\mu b N_m v(\tau),$$
$$\tau = -\frac{3}{2}(\rho - \rho_0)(u_0^2 - c^2). \tag{9}$$

Assuming that $v(\tau) = \alpha\tau$, with small values of τ we obtain $\rho - \rho_0 = e^{Ax}$, where $A = \frac{3}{2}\alpha \frac{(u_0^2 - c^2)\mu b N_m}{(c_l^2 - u_0^2) u_0} \rho_0 (c_l^2 - c^2)$.

This solution corresponds to the case where the leading front of the wave departs to infinity $(x \to -\infty)$. Under these circumstances, a steady-state profile is formed, corresponding to the asymptotic $t \to \infty$. In actuality, the time required to attain the asymptotic is determined by the value of

$$t_{as} = 1/A(c_l - u_0) = \frac{2}{3} \frac{(c_l + u_0) u_0 \rho_0 (c_l^2 - c^2)}{\alpha \mu b N_m (u_0^2 - c^2)}.$$

Thus, stationary shock waves can exist only with $u_0 > c_l$. With $u_0 \leq c_l$, the wave is found to be non-stationary and goes over to a steady-state asymptotic with $t \gg t_{as}$.

Let us examine the structure of the shock wave without taking account of the mobile dislocations with $N_m = \text{const}$. In this case, the concrete form of the equation of state must be given. We take the equation of state in the form

$$p = (\rho - \rho_0)c^2, \tag{10}$$

where c^2 is a constant. We shall also assume a constant longitudinal velocity of sound c_l.

In this case the conditions at the elastic discontinuity assume the form

$$\rho_e = \rho_0 \frac{u_0^2}{c_l^2}, \quad p_e = \frac{\rho_0 c^2}{c_l^2}(u_0^2 - c_l^2), \quad u_e = c_l^2/u_0, \quad \tau_e = -\frac{3}{2}\rho_0(c_l^2 - c^2)\left(\frac{u_0^2}{c_l^2} - 1\right).$$

Behind the front of the wave of the load, relationships corresponding to the hydrodynamic adiabatic are satisfied:

$$\rho_h = \rho_0 u_0^2/c^2, \quad p_h = \rho_0 c^2 (u_0^2/c^2 - 1), \quad u_h = c^2/u_0. \tag{11}$$

The structure of the front of the wave is conveniently investigated in the dimensionless variables

$$\xi = \mu b N_m (1 - c^2/c_l^2) x, \quad \tilde{\rho} = \rho/\rho_0, \quad \tilde{u}_0 = u_0/c_l,$$
$$\tilde{v} = v/c_l, \quad \tilde{\tau} = \frac{2}{3}\tau/\rho_0 c^2. \tag{12}$$

Equation (5) in these variables assumes the form

$$\frac{d\tilde{\rho}}{d\xi} = -\frac{\tilde{\rho}^4}{\tilde{\rho}^2 - \tilde{u}_0^2} \frac{\tilde{v}(\tilde{\tau})}{\tilde{u}_0}, \quad \tilde{\tau} = (\tilde{\rho} - 1)\left(1 - \frac{\tilde{u}_0^2 \nu^2}{\tilde{\rho}}\right), \quad \nu^2 = c_l^2/c^2. \tag{13}$$

To determine the special characteristics of the behavior of $\tilde{\rho}(\xi)$, we calculate the second derivative $d^2\tilde{\rho}/d\xi^2$

$$\frac{d^2\tilde{\rho}}{d\xi^2} = \left[2\frac{\tilde{\rho}^2 - 2\tilde{u}_0^2}{\tilde{\rho}\,(\tilde{\rho}^2 - \tilde{u}_0^2)} + \left(1 - \frac{\tilde{u}_0^2 \nu^2}{\tilde{\rho}^2}\right) \frac{v'(\tau)}{v(\tau)} \right] \left(\frac{d\tilde{\rho}}{d\xi}\right)^2. \quad (14)$$

As can be seen from this expression, the sign of the second derivative is determined by the sign of the expression standing in square brackets, and depends essentially on the behavior of the velocity of the dislocations $v(\tau)$. The experimental data on the dependence of the velocity of the dislocations on the stress is generally described by dependences of the form $v \sim \tau^n$ or $v = v_0 \exp(-\tau_0/|\tau|)$. The second dependence correctly reproduces the limiting velocity of the dislocations with large values of $|\tau|$.

Let us examine first the dependence of the form $v \sim \tau^n$. Substituting $v'(\tau)/v(\tau) = n/\tau$ into (14), we obtain

$$\frac{d^2\tilde{\rho}}{d\xi^2} = \frac{1}{\tilde{\rho}}\left(\frac{d\tilde{\rho}}{d\xi}\right)^2 \left[2\frac{\tilde{\rho}^2 - 2\tilde{u}_0^2}{\tilde{\rho}^2 - \tilde{u}_0^2} + n\frac{\tilde{\rho}^2 - \tilde{u}_0^2\nu^2}{(\tilde{\rho} - 1)(\tilde{\rho} - \tilde{u}_0^2\nu^2)} \right]. \quad (15)$$

At an elastic discontinuity $\tilde{\rho} = \tilde{u}_0^2$ and the sign of $d^2\tilde{\rho}/d\xi^2$ is determined by the sign of the quantity

$$(\tilde{u}_0^2 - u_c^2)\,[2(\nu^2 - 1) - n], \quad u_c^2 = \frac{4(\nu^2 - 1) - n\nu^2}{2(\nu^2 - 1) - n}.$$

From these relationships it can be seen that the structure of the front of the wave depends essentially on the degree of nonlinearity of the relaxation of the tangential stresses. With a weak nonlinearity, where $n < 2(\nu^2 - 1)$, for strong shock waves ($\tilde{u}_0^2 > u_c^2$) the second derivative is found to be positive at the elastic discontinuity. We note that, if $n < 2$, then $u_c > 1$ and, for sufficiently weak shock waves ($\tilde{u}_0 > u_c$) the second derivative is negative. In the case of strong nonlinearity, where $n > 2(\nu^2 - 1)$, the situation changes. If, in this case, the condition $n > 2$ is satisfied, then $u_c > 1$ and the second derivative is positive only for rather weak shock waves ($\tilde{u}_0 < u_c$). For strong waves ($\tilde{u}_0 > u_c$), the second derivative near an elastic discontinuity is found to be negative. This kind of behavior of the profile of the wave is connected with the sharp dependence of the velocity of the dislocations on the tangential stresses. As follows from expression (13), maximal tangential stresses are attained at the line $\tilde{\rho} = \tilde{u}_0\nu$, which corresponds to curve 4 on Fig. 1, intersecting the line of the shock front at the point $\tilde{u}_1 = \nu$. It can be shown that the condition $\tilde{u}_1 < u_c$ is always satisfied. As can be seen from expression (15), with $\tilde{\rho} \to \tilde{u}_0^2\nu^2$ the sign of the second derivative is negative (since $\tilde{\rho} < \tilde{u}_0^2\nu^2$) with any values of n. Therefore, in the dependence on the velocity of the shock wave and the degree of nonlinearity of the relaxation of the tangential stresses, the following characteristic profiles of the shock wave can be observed: a) a weak nonlinearity ($n < 2(\nu^2 - 1)$) and weak waves ($u_0^2 < u_c^2$): either an ordinary relaxation profile with a negative second derivative, shown qualitatively in Fig. 2 (curve 1), or the profile shown by curve 2 in Fig. 2; b) a weak nonlinearity ($n < 2(\nu^2 - 1)$) and strong waves ($\tilde{u}_0^2 > u_c^2$): the profile shown qualitatively by curve 3 in Fig. 2; c) a strong non-linearity ($n > 2(\nu^2 = 1)$) and weak shock waves ($\tilde{u}_0^2 < u_c^2$): the profile shown qualitatively by curve 3 in Fig. 2; d) a strong non-linearity ($n > 2(\nu^2 - 1)$) and strong shock waves ($\tilde{u}_0^2 > u_c^2$): the relaxation profile shown qualitatively by curve 1 in Fig. 2; the presence of a point of inflection at the front of the shock wave in cases b) and c) has a different nature. While in case b) it arises due to the kinematics, in case c) the point of inflection arises due to the development of plastic flow. Therefore, only in case c) is it possible to identify it with the plastic wave described in [7].

Let us consider briefly the dependence $v = v_0 \exp(-\tau_0/|\tau|)$. In this case, in (15) we must substitute $n_{eff} = \tau_0/|\tau|$. At an elastic discontinuity $n_{eff} = 2\tau_0/3\rho_0 c^2 (\nu^2 - 1)(\tilde{u}_0 - 1)$. As can be seen from this expression, only very weak waves ($\tilde{u}_0 - 1 \ll 1$) will be characterized by large values of n_{eff}. For the remaining waves $n_{eff} \ll 1$, since, according to the experimental data, $\tau_0 \ll \rho_0 c^2$. From expression (15) it can be seen that the profile of waves with $\tilde{u}_0 < 2$ will have the form given in Fig. 2 (curve 1). This conclusion is confirmed by a numerical calculation. Figure 3a shows profiles of waves obtained by the numerical integration of Eq. (13) for the following values of the parameters: $\nu^2 = 1.7$, $\tau_0/\rho_0 c^2 = 2 \cdot 10^{-3}$, $\tilde{u}_{01} = 1.001$, $\tilde{u}_{02} = 1.05$, $\tilde{u}_{03} = 1.1$, curves 1-3, respectively.

Let us evaluate the width of the shock front. Using Eq. (13) and taking into consideration that $\tilde{v} < 1$, $\nu^2 \sim 1$, and $\tilde{u}_0 \sim 1$, we obtain

$$\Delta\xi_f \sim 1. \quad (16)$$

Thus, going over to dimensional values, and taking $\mu b N_m \sim 1\ \text{cm}^{-1}$, we obtain $\Delta \sim 1$ cm.

Let us examine the structure of the wave with a varying value of N_m. During the process of plastic deformation, there is a change in the density of the mobile dislocations, which manifests itself as deformation hardening. This effect is taken into consideration in Eq. (5), in which the density of the mobile dislocations depends on the value of the plastic deformation. Using the equation of state (10) and solving Eq. (6), we obtain

$$\gamma_p(\rho) = \frac{\nu^2}{\nu^2 - 1}\left(\ln\frac{\tilde{\rho}}{\tilde{\rho}_e} + \frac{1}{2}\tilde{u}_0^2 \frac{\tilde{\rho}_e^2 - \tilde{\rho}^2}{\tilde{\rho}^2\tilde{\rho}_e^2} \right). \quad (17)$$

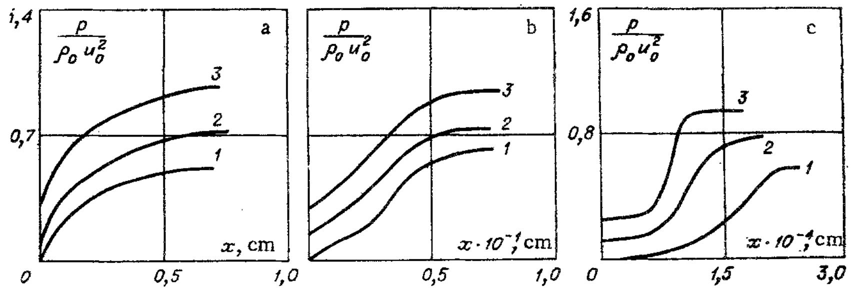

Fig. 3

In this expression it is taken into account that, at an elastic discontinuity, there is no plastic deformation. A simple form of Eq. (17) is generally taken near the elastic discontinuity, where $(\tilde{\rho} - \tilde{\rho}_e)/\tilde{\rho}_e \ll 1$,

$$\gamma_p = \frac{\nu^2}{\nu^2-1}\frac{\tilde{u}_0^2-1}{\tilde{u}_0^4}(\tilde{\rho} - \tilde{u}_0^2); \tag{18}$$

here it is taken into consideration that $\tilde{\rho}_e = \tilde{u}_0^2$. If the condition $\nu^2 - 1 \ll 1$ is satisfied, then expression (18) is valid at the whole front of the shock wave. We note that relationships (17) and (18) make it possible to determine the maximal plastic deformation taking place in the front of the shock wave. Substituting relationship (11) into expression (17), we obtain

$$\gamma_p^{\max} = \frac{\nu^2}{\nu^2-1}\left[\ln \nu^2 - \frac{1}{2\tilde{u}_0^2}(1-\nu^{-4})\right]. \tag{19}$$

if $\nu^2 - 1 \ll 1$, then (19) gives $\gamma_p^{\max} = \nu^2(1-\tilde{u}_0^{-2})$. As can be seen from (19), the value of $\gamma_p^{\max}$ rises monotonically with a rise in $\tilde{u}_0$ and ν^2. Expression (19) makes it possible to determine the density of the dislocations behind the front of a shock wave with a known dependence $N(\gamma_p)$.

To investigate the effect of a change in the density of the mobile dislocations on the structure of the shock wave, we postulate that the density of the mobile dislocations is determined by the law [1, 4, 5]:

$$N_m = N_{m0}(1 + m\gamma_p), \tag{20}$$

where m is a coefficient determining the change in the density of the dislocations. In the case $m < 0$, this law reflects a linear hardening of the material, and $m > 0$ a weakening. Using (20), Eq. (5) in dimensionless variables (12) assumes the form

$$\frac{d\tilde{\rho}}{d\xi} = -\frac{\tilde{\rho}^4(1+m\gamma_p)}{\tilde{\rho}^2 - \tilde{u}_0^2}\frac{\tilde{v}(\tilde{\tau})}{\tilde{u}_0},$$

where $\xi = \mu b N_{m0}[1-(c^2/c_l^2)]x$, and $\gamma_p(\tilde{\rho})$ is given by expression (17). Multiplication of the dislocations leads to the appearance of a distinct plastic front and to a decrease in the width of the front of the shock wave. Evaluating the width of the front in a way analogous to that used in the derivation of (16), we obtain

$$\Delta\xi_f \sim (1 + m\gamma_p^{\max})^{-1}. \tag{21}$$

From this it can be seen that taking account of the multiplication of the dislocations leads to a stronger dependence of the width of the front on the velocity of the shock wave. The simplest form of expression (21) is taken with $\nu^2 - 1 \ll 1$, where $1 + m\gamma_p^{\max} \simeq 1 + m\nu^2(1-\tilde{u}_0^{-2})$. In this case, with $m \gg 1$, we obtain

$$\Delta\xi_f \sim \frac{1}{m\nu^2}\frac{\tilde{u}_0^2}{\tilde{u}_0^2 - 1}.$$

From this it can be seen that for weak shock waves $(\tilde{u}_0^2 - 1 \ll 1)$, the width of the front will depend greatly on the power of the wave. With $\tilde{u}_0^2 \gg 1$, this dependence becomes unreal.

The value of m is determined from experiments on plastic deformation and is found equal to $\sim 10^5$ [1, 5, 6]. Substituting $\nu^2 = 2$ into (19) and evaluating $\Delta\xi_f$ using formula (21), we obtain $\Delta\xi_f \sim 10^{-4}$ with $m \sim 10^5$ and $\Delta\xi_f \sim 10^{-1}$ with $m \sim 10^2$. This evaluation is confirmed by a numerical calculation, shown in Fig. 3b, c; the calculations were made for the following values of the parameters: $\nu^2 = 1.7$, $\tau_0/\rho_0 c^2 = 2 \cdot 10^{-3}$, $\tilde{u}_{01} = 1.001$, $\tilde{u}_{02} = 1.05$, $\tilde{u}_{03} = 1.1$ (curves 1-3, respectively), $m = 10^2$, and $m = 10^5$ (Fig. 3b, c, respectively).

Using relationships (19), (20), the density of the dislocations behind the front of the shock wave can be evaluated. A maximal value of the density is attained with strong shock waves $(\tilde{u}_0^2 \gg 1)$ and is equal to $N_m = N_0[1 + m\nu^2 \ln(\nu^2/\nu^2 - 1)]$. With $\nu^2 \sim 2$, $N_0 \sim 10^7$ cm^{-2}, and $m \sim 10^5$, the maximal density of the dislocations is $\sim 10^{12}$ cm^{-2}.

Let us examine the asymptotic structure of a wave with $u_0 \leq c_l$ without taking account of the change in the density of the mobile dislocations. As is shown by an analysis of the second derivative, given by expression (14), it reverts to zero only at one point. Thus, with $u_0 \leq c_l$, there should be one point of inflection at the front of the wave. We note that, with a rise in the value of n (strong nonlinearity of the relaxation rate), this point of inflection approaches the point $\tilde{\rho} = \tilde{u}_0 \nu$, which corresponds to maximal tangential stresses. Thus, nonlinearity of the relaxation rate leads to a situation in which the greatest curvature of the front of the wave is determined by the point of a maximal relaxation rate.

The structure of the leading section of the front is determined by Eq. (9), which, in dimensionless variables (12) for $v(\tau) = \alpha\tau^n$ assumes the form

$$\frac{d\tilde{\rho}}{d\xi} = \frac{\tilde{\alpha}\left(\tilde{u}_0^2 \nu^2 - 1\right)^n}{(\tilde{u}_0 - 1)\,\tilde{u}_0} \frac{(\tilde{\rho} - 1)^n}{(\tilde{\rho} - \tilde{u}_0)} \equiv A\left(\tilde{u}_0\right) \frac{(\tilde{\rho} - 1)^n}{(\tilde{\rho} - \tilde{u}_0)}, \tag{22}$$

where $\tilde{\alpha} = \alpha(3\rho_0 c^2/2)^n / c_l$. As can be seen, the character of the rise in the stresses depends on the value of n. In addition, the structure of the leading section will differ with $\tilde{u}_0 = 1$ and $\tilde{u}_0 < 1$.

The structure of the rear part is determined by Eq. (13) with $\tilde{\rho} \to u_0^2 \nu^2$ which, for $v(\tau) = \alpha\tau^n$ assumes the form

$$\frac{d\tilde{\rho}}{d\xi} = \frac{\tilde{\alpha}\left(\tilde{u}_0^2 \nu^2 - 1\right)^n}{(\nu^2 - 1)\,\nu^{2(n-4)}\,\tilde{u}_0^{2n-5}} \left(\tilde{u}_0^2 \nu^2 - \tilde{\rho}\right)^n \equiv B\left(\tilde{u}_0\right)\left(\tilde{u}_0^2 \nu^2 - \tilde{\rho}\right)^n.$$

The relaxation of the density to the hydrostatic value behind the front of the wave depends essentially on the value of n. We note that the value $\tilde{u}_0 = 1$ is not isolated with a consideration of the structure of the rear section of the front.

Let us first consider the case $\tilde{u}_0 = 1$ ($u_0 = c_l$). The solution of Eq. (22) with $n < 2$ will have the form $\tilde{\rho} - 1 = [(2-n)A(1)\xi]^{1/(2-n)}$; the front of such a wave has a finite extension. With $n \leq 1$, the leading front will be a weak discontinuity. Here, for $n < 1$ (this case is obviously not realized in practice, since it corresponds to an increase in the relaxation time with a rise in the tangential stresses), the derivative at the leading front reverts to infinity. With $n = 2$, the solution of Eq. (22) will have the form $\tilde{\rho} - 1 = \exp\{A(1)\xi\}$ and, with $n > 2$, $\tilde{\rho} - 1 = [(2-n)A(1)\xi]^{1/(n-2)}$. Thus, with a sufficiently strong nonlinearity ($n \geq 2$), a wave moving with the velocity $u_0 = c_l$ becomes a running wave and asymptotically approaches a form where the rise at the leading front takes place exponentially ($n = 2$) or according to a power law ($n > 2$).

For $\tilde{u}_0 < 1$, the structure of the leading section with $n < 1$ will be given by the relationship $\tilde{\rho} - 1 = [\xi(1 - n)A(\tilde{u}_0)/(1 - \tilde{u}_0)]^{1/(1-n)}$; in this case, the expansion of the wave remains finite. This is connected with the fact that small stresses relax more rapidly and cannot depart from the large stresses. With $n = 1$, the rise in the density at the leading section has an exponential character $\tilde{\rho} - 1 = \exp\{A(\tilde{u}_0)\xi/(1 - \tilde{u}_0)\}$. Under these circumstances, with a decrease in $\tilde{u}_0$, the value of $A(\tilde{u}_0)/(1 - \tilde{u}_0)$ decreases and, at the limit $\tilde{u}_0 \to 1/\nu$, reverts to zero. With $n > 1$, the density in the leading section varies in accordance with a power law: $\rho - 1 = [(1-n)A(\tilde{u}_0)\xi/(1 - \tilde{u}_0)]^{1/(n-1)}$.

Let us now examine the character of the rear section of the wave. With $n < 1$, the density relaxes to its hydrostatic value $\tilde{\rho} = \tilde{u}_0 \nu^2$ in a finite time. This is connected with the fact that small stresses relax more rapidly than large. With $n = 1$, the hydrostatic value is attained in an exponential manner: $\tilde{u}_0^2 \nu^2 - \tilde{\rho} = \exp\left[-B\left(\tilde{u}_0\right)\xi\right]$, and, with $n > 1$, according to a power law: $\tilde{u}_0^2 \nu^2 - \tilde{\rho} = \left[(n-1)\,B\left(\tilde{u}_0\right)\xi\right]^{-1/(n-1)}$.

We note that, for any given values of n, the leading section of the wave with $\tilde{u}_0 > 1/\nu$ ($u_0 > c$) is found to be steeper than the rear section, and the profile of the wave is asymmetrical. With a decrease in $\tilde{u}_0$ this asymmetry decreases and, at the limit of very weak waves ($\tilde{u}_0 \to 1/\nu$), the profile of the wave becomes symmetrical.

Let us examine in more detail the structure of a weak wave, whose profile asymptotically approaches the solution of Eq. (13) with $\tilde{u}_0 \to 1/\nu$ ($u_0 \to c$). Expanding this equation near the point $\tilde{\rho} = 1$ and taking into account that $\tilde{u}_0 \nu \to 1$, we obtain

$$\frac{d\tilde{\rho}}{d\xi} = \frac{\tilde{\alpha}\nu^3}{\nu^2 - 1}(\tilde{\rho} - 1)^n \left(\tilde{u}_0^2 \nu^2 - \tilde{\rho}\right)^n.$$

With $n = 1$, the solution of this equation has the form

$$\tilde{\rho} = \frac{1}{2}\left\{\left(\tilde{u}_0^2 \nu^2 + 1\right) + \left(\tilde{u}_0^2 \nu^2 - 1\right) \operatorname{th} C\xi\right\},$$

where $C = \tilde{\alpha}\nu^3(\tilde{u}_0^2 \nu^2 - 1)/(\nu^2 - 1)$. This solution is well known from problems of nonlinear acoustics [10]. With $n > 1$, the profile of the wave remains symmetrical, but the dependence of the curvature of the front of the velocity of the wave becomes sharper: $d\rho/d\xi|_{\max} \sim (\tilde{u}_0^2 v^2 - 1)^n$. We note that, for a weak wave, taking account of the change in the density of the mobile dislocations gives corrections of the following order of smallness with respect to $(\tilde{u}_0^2 v^2 - 1)$.

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INSTABILITY OF A SPHERICAL BODY UNDER UNIFORM LOADING

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Proceeding from the three-dimensional equations of stability theory in the dynamical formulation, the stability of a sphere made out of a reinforced elastic–viscous–plastic material is investigated under uniform loading. The subcritical strains are small. It is shown that the results obtained from approximate and three-dimensional theories for elastic–plastic stability problems differ qualitatively and quantitatively in practice. A similar problem has been discussed earlier in [1] in a static formulation on the basis of an approximate approach and the relationships of the theory of small elastic–plastic strains.

The axisymmetric elastic–plastic state of a spherical body of radii r_1 and r_2 subject to the action of an internal pressure q is determined by the relationships

$$\begin{aligned}
\sigma_r^{p0} &= -q_0 + \frac{4k_0}{2+c_0}\left[6\ln\frac{r}{\alpha} + c_0\gamma^3\left(\frac{1}{\alpha^3} - \frac{1}{r^3}\right)\right],\\
\sigma_\theta^{p0} &= -q_0 + \frac{4k_0}{2+c_0}\left[6\ln\frac{r}{\alpha} + 3 + c_0\gamma^3\left(\frac{1}{\alpha^3} + \frac{1}{2r^3}\right)\right],\\
\sigma_r^{e0} &= 4k_0\gamma^3\left(1 - \frac{1}{r^3}\right) \quad \sigma_\theta^{e0} = 4k_0\gamma^3\left(1 + \frac{1}{2r^3}\right),\\
\alpha &= r_1 r_2^{-1}.
\end{aligned} \tag{1}$$

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